

Global quantum correlations in tripartite nonorthogonal states and monogamy properties

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Abstract

A global measure of quantum correlations for tripartite nonorthogonal states is presented. It is introduced as the overall average of the pairwise correlations existing in all possible partitions. The explicit expressions for the global measure are derived for squared concurrence, entanglement of formation, quantum discord and its geometric variant. As illustration, we consider even and odd three-mode Schrödinger cat states based on Glauber coherent states. We also discuss limitations to sharing quantum correlations known as monogamy relations.

1 Introduction and motivations

Remarkable achievements in characterizing, identifying and quantifying quantum correlations in bipartite quantum systems were accomplished in the last two decades [1, 2, 3, 4, 5] (for a recent review see [6]). Quantum entanglement is an useful resource for quantum information processing such as quantum teleportation [7], superdense coding [8], quantum key distribution [9], telecloning [10] and many more. Until some time ago, entanglement was usually regarded as synonymous of quantum correlation and subsequently considered as the only type of nonclassical existing in a multipartite quantum system. However, quantum entanglement does not account for all nonclassical aspects of quantum correlations and unentangled mixed states can possess quantum correlations. In this respect, other measures of quantum correlations beyond entanglement were studied. The most popular among them is quantum discord introduced in [11, 12]. It coincides with entanglement of formation for pure states. For mixed states, the explicit evaluation of quantum discord involves potentially complex optimization procedure which was achieved for a limited set of two qubit systems [13, 14, 15, 16, 17, 18, 19]. To overcome this problem an alternative geometrized variant of quantum discord was introduced [20]. Nowadays, entanglement of formation [21], quantum discord [11, 12] and its geometric variant [20] are typical examples of bipartite measures commonly used to decide about the presence of quantum correlations in a bipartite quantum system.

In other hand, the characterization of genuine correlations in multipartite quantum systems encounters many conceptual obstacles and the extension of usual bipartite measures for many-particles systems is not well understood [6]. Despite many efforts regarding this problem [22, 23, 24, 25, 26], there are still many unsolved issues. The main motivation behind these efforts relies upon the recent experimental results reporting the creation and manipulation of macroscopic quantum states and highly correlated atomic ensembles such as spin squeezed states [27, 28, 29]. Accordingly, different approaches to quantify multipartite correlations in quantum systems have been proposed in the literature [30, 31, 32]. In particular, Rulli and Sarandy [31] defined the multipartite measure of quantum correlation as the maximum of the quantum correlation existing between all possible bipartition of the multipartite quantum system. In this paper, paralleling the treatment discussed in [32], we define the global quantum correlation present in a tripartite system ABC of type (3) as the sum of the correlations of all possible bi-partitions. Explicitly, it is given by

$$Q_{(A,B,C)} = \frac{1}{12} \left(Q_{AB} + Q_{BA} + Q_{AC} + Q_{CA} + Q_{BC} + Q_{CB} \right. \\ \left. + Q_{A(BC)} + Q_{(BC)A} + Q_{B(AC)} + Q_{(AC)B} + Q_{C(AB)} + Q_{(AB)C} \right) \quad (1)$$

where the measure Q stands for concurrence, entanglement of formation, entropy based quantum discord or geometric quantum discord.

Another important feature appearing in investigating multipartite quantum correlations is the

so-called monogamy relation which imposes severe restriction of shareability of quantum correlations in a quantum system comprising three or more parts. The monogamy relation was first considered by Coffman, Kundo and Wootters in 2001 [33] in analyzing the distribution of entanglement in a tripartite qubit system. Since then, the monogamy relation was extended to other measures of quantum correlations. Unlike the squared concurrence [33], the entanglement of formation do not satisfy the monogamy relation [33] in a pure tripartite qubit system but it is satisfied in multi-mode Gaussian state [34, 35]. Furthermore, quantum correlations, measured by quantum discord, were shown to violate monogamy in some specific quantum states [36, 37, 38, 39, 40]. Now, there are many attempts to establish the general conditions under which a given quantum correlation measure is monogamous or not (see [41] and references quoted therein). The concept of monogamy can be summarized as follows. Let Q_{AB} denote the shared correlation Q between A and B . Similarly, let us denote by Q_{AC} the measure of correlation between A and C and $Q_{A(BC)}$ the correlation shared between A and the composite subsystem BC comprising B and C . The measure Q is monogamous if and only if the following quantity

$$Q_{A|BC} = Q_{A(BC)} - Q_{AB} - Q_{AC} \quad (2)$$

is positive. Therefore, quantifying the global correlation and analyzing the monogamy of the measure Q can be obtained by quantifying pairwise correlations among subsystems.

In this work, we derive the global quantum correlations in pure tripartite nonorthogonal states based on the sum of correlations for all possible bi-partitions. This is done for the widely-used measures: concurrence, entanglement of formation, quantum discord and geometric quantum discord. To convert the nonorthogonal states to qubits, a qubit mapping is realized. This realization is similar to one recently used in the analysis of bipartite entanglement properties in bipartite coherent states [18, 19, 42, 43, 44, 45]. As special instance of superpositions of nonorthogonal states, we consider three-mode Schrödinger cat states, based on Glauber coherent states. We give the explicit expressions of the global tripartite correlations. We also discuss the limitations to sharing quantum correlations.

This paper is organized as follows. In order to discuss the pairwise quantum correlations in entangled tripartite nonorthogonal states, we introduce, in Section 2, two different partitioning schemes. For each scheme, a qubit mapping is proposed. In section 3, we give the analytic expressions of pairwise entanglement of formation and quantum discord. We discuss the conservation relation between these two entropy based measures which implies that the tripartite measure for quantum discord and the entanglement of formation are identical. In section 4, we derive the geometric quantum discord for all possible bipartite subsystems. As illustration, we consider in section 5, three-mode Schrödinger cat states, based on Glauber coherent states. In particular, we discuss the monogamy property of entanglement measured by concurrence, entanglement of formation, quantum discord and geometric quantum discord. Concluding remarks close this paper.

2 Tripartite nonorthogonal states

Usually, a tripartite state shared between three parties A , B and C is designated by a unit-trace bounded operator ρ_{ABC} . In this work, we shall consider the pure tripartite state comprising three identical subsystems living in the Hilbert space $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ where \mathcal{H} is spanned by the set of orthonormal vectors $\{|e_n\rangle : n = 1, 2, \dots, d\}$. The dimension d of \mathcal{H} may be either finite or infinite. To simplify further our purpose, we focus on tripartite balanced entangled state of the form

$$|\Psi, m\rangle = \mathcal{N}(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle + e^{im\pi} |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle) \quad (3)$$

where $m \in \mathbf{Z}$, $|\psi_i\rangle$ and $|\phi_i\rangle$ are normalized states of the subsystem i ($i = 1, 2, 3$). They are linear superpositions of the eigenstates $\{|e_n\rangle\}$ of the subsystem i . The overlaps $\langle\psi_i|\phi_i\rangle = p_i$ are in general non zero. In the equation (3), \mathcal{N} is given by

$$\mathcal{N} = [2 + 2p_1p_2p_3 \cos m\pi]^{-1/2}$$

and stands for the normalization factor of the tripartite state $|\Psi, m\rangle$. We assume that p_1 , p_2 and p_3 are reals. Typical examples of nonorthogonal entangled of the form (3) are the superpositions of coherent and squeezed states. As mentioned in the introduction, to determine the explicit expressions of pairwise quantum correlations present in (3), the whole system can be partitioned in two different ways. For each bipartition, the bipartite states are mapped into a two qubit systems passing from nonorthogonal states to an orthonormal basis. This technique is similar to one used in [46, 47, 48, 49] to investigate entanglement properties for multipartite coherent states.

2.1 Pure bi-partitions and qubit mapping

We first consider pure bipartite splitting of the tripartite system (3). In this case, the entire system splits into two subsystems, one subsystem containing one particle and the other containing the remaining particles. Three partitions are possible. Indeed, the state $|\Psi, m\rangle$ can be decomposed as

$$|\Psi, m\rangle = \mathcal{N}(|\psi\rangle_k \otimes |\psi\rangle_{ij} + e^{im\pi} |\phi\rangle_k \otimes |\phi\rangle_{ij}) \quad (4)$$

where

$$|\psi\rangle_k = |\psi_k\rangle, \quad |\phi\rangle_k = |\phi_k\rangle \quad k = 1, 2 \text{ or } 3,$$

and

$$|\psi\rangle_{ij} = |\psi\rangle_i \otimes |\psi\rangle_j \quad i, j \neq k$$

is the state describing the modes i and j . The three particles state $|\Psi, m\rangle$ can be expressed by means of two logical qubits. This can be realized as follows. We introduce, for the first subsystem, the orthogonal basis $\{|0\rangle_k, |1\rangle_k\}$ defined by

$$|0\rangle_k = \frac{|\psi\rangle_k + |\phi\rangle_k}{\sqrt{2(1+p_k)}} \quad |1\rangle_k = \frac{|\psi\rangle_k - |\phi\rangle_k}{\sqrt{2(1-p_k)}}. \quad (5)$$

Similarly, we introduce, for the second subsystem (ij), the orthogonal basis $\{|0\rangle_{ij}, |1\rangle_{ij}\}$ given by

$$|0\rangle_{ij} = \frac{|\psi\rangle_{ij} + |\phi\rangle_{ij}}{\sqrt{2(1 + p_i p_j)}} \quad |1\rangle_{ij} = \frac{|\psi\rangle_{ij} - |\phi\rangle_{ij}}{\sqrt{2(1 - p_i p_j)}}. \quad (6)$$

Inserting (5) and (6) in (4), we get the form of the pure state $|\Psi, m\rangle$ in the basis $\{|0\rangle_k \otimes |0\rangle_{ij}, |0\rangle_k \otimes |1\rangle_{ij}, |1\rangle_k \otimes |0\rangle_{ij}, |1\rangle_k \otimes |1\rangle_{ij}\}$. Explicitly, it is given by

$$|\Psi, m\rangle = \sum_{\alpha=0,1} \sum_{\beta=0,1} C_{\alpha,\beta} |\alpha\rangle_k \otimes |\beta\rangle_{ij} \quad (7)$$

where the coefficients $C_{\alpha,\beta}$ are

$$\begin{aligned} C_{0,0} &= \mathcal{N}(1 + e^{im\pi}) c_k^+ c_{ij}^+, & C_{0,1} &= \mathcal{N}(1 - e^{im\pi}) a_k^+ c_{ij}^- \\ C_{1,0} &= \mathcal{N}(1 - e^{im\pi}) c_{ij}^+ c_k^-, & C_{1,1} &= \mathcal{N}(1 + e^{im\pi}) c_k^- c_{ij}^-. \end{aligned}$$

in terms of the quantities

$$c_k^\pm = \sqrt{\frac{1 \pm p_k}{2}} \quad c_{ij}^\pm = \sqrt{\frac{1 \pm p_i p_j}{2}}$$

involving the scalar products p_i between the nonorthogonal states $|\psi_i\rangle$ and $|\phi_i\rangle$.

2.2 Mixed bi-partitions and qubit mapping

The second partition can be realized by considering the bipartite reduced density matrix ρ_{ij} which is obtained by tracing out the degrees of freedom of the third subsystem k :

$$\rho_{ij} = \text{Tr}_{k \neq i,j} (|\Psi, m\rangle \langle \Psi, m|). \quad (8)$$

In this case, three different bipartite mixed states are also possible: ρ_{12} , ρ_{13} and ρ_{23} . The reduced density matrix ρ_{ij} is given by

$$\rho_{ij} = \mathcal{N}^2 (|\psi_i, \psi_j\rangle \langle \psi_i, \psi_j| + |\phi_i, \phi_j\rangle \langle \phi_i, \phi_j| + e^{im\pi} q_{ij} |\phi_i, \phi_j\rangle \langle \psi_i, \psi_j| + e^{-im\pi} q_{ij} |\psi_i, \psi_j\rangle \langle \phi_i, \phi_j|) \quad (9)$$

with $q_{ij} \equiv p_1 p_2 p_3 / p_i p_j$. It is interesting to note that the density ρ_{ij} is a rank-2 mixed state. Indeed, the state (9) can be written as

$$\rho_{ij} = \frac{\mathcal{N}^2}{\mathcal{N}_{ij}^2} \left[a_{ij}^2 |\Psi_{ij}\rangle \langle \Psi_{ij}| + b_{ij}^2 Z |\Psi_{ij}\rangle \langle \Psi_{ij}| Z \right] \quad (10)$$

where \mathcal{N}_{ij} is the normalization factor of the bipartite state $|\Psi_{ij}\rangle$ given by

$$|\Psi_{ij}\rangle = \mathcal{N}_{ij} (|\psi_i, \psi_j\rangle + e^{im\pi} |\phi_i, \phi_j\rangle)$$

and the operator Z is the third Pauli generator defined by

$$Z |\Psi_{ij}\rangle = \mathcal{N}_{ij} (|\psi_i, \psi_j\rangle - e^{im\pi} |\phi_i, \phi_j\rangle).$$

The coefficients a_{ij} and b_{ij} occurring in (10) are expressed in terms of the quantities q_{ij} as follows

$$a_{ij} = \sqrt{\frac{1+q_{ij}}{2}} \quad b_{ij} = \sqrt{\frac{1-q_{ij}}{2}}.$$

Here also, one can map the reduced system ρ_{ij} into a pair of two-qubits. As hereinabove, we define, for the subsystem i , the orthogonal basis $\{|\mathbf{0}_i\rangle, |\mathbf{1}_i\rangle\}$ by

$$|\psi_i\rangle \equiv a_i|\mathbf{0}_i\rangle + b_i|\mathbf{1}_i\rangle \quad |\phi_i\rangle \equiv a_i|\mathbf{0}_i\rangle - b_i|\mathbf{1}_i\rangle, \quad (11)$$

where

$$a_i = \sqrt{\frac{1+p_i}{2}} \quad b_i = \sqrt{\frac{1-p_i}{2}}.$$

Similarly, we introduce, for the subsystem j , a second two dimensional orthogonal basis as

$$|\psi_j\rangle \equiv a_j|\mathbf{0}_j\rangle + b_j|\mathbf{1}_j\rangle \quad |\phi_j\rangle \equiv a_j|\mathbf{0}_j\rangle - b_j|\mathbf{1}_j\rangle, \quad (12)$$

where

$$a_j = \sqrt{\frac{1+p_j}{2}} \quad b_j = \sqrt{\frac{1-p_j}{2}}.$$

Substituting Eqs. (11) and (12) into Eq. (9), it is simple to reexpress the 2-rank mixed density (10) in the two qubit basis $\{|\mathbf{0}_i\mathbf{0}_j\rangle, |\mathbf{0}_i\mathbf{1}_j\rangle, |\mathbf{1}_i\mathbf{0}_j\rangle, |\mathbf{1}_i\mathbf{1}_j\rangle\}$. The pure as well as mixed bi-partitions and the qubit mappings introduced in this section provides us with a simple way to derive the pairwise quantum correlations and subsequently the global quantum correlations in multipartite nonorthogonal states. This is discussed in the following sections.

3 Quantum discord and entanglement of formation in tripartite nonorthogonal states

3.1 Bipartite measures of entanglement of formation and quantum discord

The total correlation in a quantum state ρ_{AB} is quantified by the mutual information

$$I_{AB} = S_A + S_B - S_{AB}, \quad (13)$$

where ρ_{AB} is the state of a bipartite quantum system composed of the subsystems A and B , the operator $\rho_{A(B)} = \text{Tr}_{B(A)}(\rho_{AB})$ is the reduced state of $A(B)$ and $S(\rho)$ is the von Neumann entropy of a quantum state ρ . The mutual information I_{AB} contains both quantum and classical correlations. It can be decomposed as

$$I_{AB} = D_{AB} + C_{AB}.$$

Consequently, for a bipartite quantum system, the quantum discord D_{AB} is defined as the difference between total correlation I_{AB} and classical correlation C_{AB} . The classical part C_{AB} can be determined by a local measurement optimization procedure as follows. Let us consider a perfect measurement on the subsystem A defined by a positive operator valued measure (POVM). The set of POVM elements

is denoted by $\mathcal{M} = \{M_k\}$ with $M_k \geq 0$ and $\sum_k M_k = \mathbb{I}$. The von Neumann measurement, on the subsystem A , yields the statistical ensemble $\{p_{B,k}, \rho_{B,k}\}$ such that

$$\rho_{AB} \longrightarrow \frac{(M_k \otimes \mathbb{I})\rho_{AB}(M_k \otimes \mathbb{I})}{p_{B,k}}$$

where the measurement operation is written as [13]

$$M_k = U \Pi_k U^\dagger \quad (14)$$

with $\Pi_k = |k\rangle\langle k|$ ($k = 0, 1$) is the one dimensional projector for subsystem A along the computational basis $|k\rangle$, $U \in SU(2)$ is a unitary operator and

$$p_{B,k} = \text{Tr} \left[(M_k \otimes \mathbb{I})\rho_{AB}(M_k \otimes \mathbb{I}) \right].$$

The amount of information acquired about particle B is then given by

$$S_B - \sum_k p_{B,k} S_{B,k},$$

which depends on measurements belonging to \mathcal{M} . To remove the measurement dependence, a maximization over all possible measurements is performed and the classical correlation writes

$$\begin{aligned} C_{AB} &= \max_{\mathcal{M}} \left[S_B - \sum_k p_{B,k} S_{B,k} \right] \\ &= S(\rho_B) - \tilde{S}_{\min} \end{aligned} \quad (15)$$

where \tilde{S}_{\min} denotes the minimal value of the conditional entropy

$$\tilde{S} = \sum_k p_{B,k} S_{B,k}. \quad (16)$$

When optimization is taken over all perfect measurement, the quantum discord is

$$D_{AB} \equiv D_{AB}^{\rightarrow} = I_{AB} - C_{AB} = S_A + \tilde{S}_{\min} - S_{AB}. \quad (17)$$

Thus, the derivation of quantum discord requires the minimization of conditional entropy. This constitutes a complicated issue when dealing with an arbitrary mixed state. The explicit analytical expressions of quantum discord were obtained only for few exceptional two-qubit quantum states, especially ones of rank two. One may quote for instance the results obtained in [14, 34] (see also [18, 19, 45]). For a density matrix of rank two, the minimization of the conditional entropy (16) can be performed by purifying the density matrix ρ_{AB} and making use of Koashi-Winter relation [50] (see also [15]). This relation establishes the connection between the classical correlation of a bipartite state ρ_{AB} and the entanglement of formation of its complement ρ_{BC} . Hereafter, we discuss briefly this nice relation. For a rank-two quantum state, the density matrix ρ_{AB} decomposes as

$$\rho_{AB} = \lambda_+ |\phi_+\rangle\langle\phi_+| + \lambda_- |\phi_-\rangle\langle\phi_-| \quad (18)$$

where λ_+ and λ_- are the eigenvalues of ρ_{AB} and the corresponding eigenstates are denoted by $|\phi_+\rangle$ and $|\phi_-\rangle$ respectively. Attaching a qubit C to the two-qubit system A and B , the purification of the system yields

$$|\phi\rangle = \sqrt{\lambda_+}|\phi_+\rangle \otimes |\mathbf{0}\rangle + \sqrt{\lambda_-}|\phi_-\rangle \otimes |\mathbf{1}\rangle \quad (19)$$

such that the whole system ABC is described by the pure state $\rho_{ABC} = |\phi\rangle\langle\phi|$ from which one has the bipartite densities $\rho_{AB} = \text{Tr}_C \rho_{ABC}$ and $\rho_{BC} = \text{Tr}_A \rho_{ABC}$. According to Koashi-Winter relation [50], the minimal value of the conditional entropy coincides with the entanglement of formation of ρ_{BC} . It is given by

$$\tilde{S}_{\min} = E(\rho_{BC}) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |\mathcal{C}(\rho_{BC})|^2}\right) \quad (20)$$

where $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function and $\mathcal{C}(\rho_{BC})$ is the concurrence of the density ρ_{BC} . We recall that for ρ_{12} the density matrix for a pair of qubits 1 and 2 which may be pure or mixed, the concurrence is [21]

$$\mathcal{C}_{12} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \quad (21)$$

for $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ the square roots of the eigenvalues of the "spin-flipped" density matrix

$$\varrho_{12} \equiv \rho_{12}(\sigma_y \otimes \sigma_y) \rho_{12}^*(\sigma_y \otimes \sigma_y), \quad (22)$$

where the star stands for complex conjugation in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and σ_y is the usual Pauli matrix. It follows that the Koashi-Winter relation and the purification procedure provide us with a computable expression of quantum discord

$$D_{AB}^{\rightarrow} = S_A - S_{AB} + E_{BC} \quad (23)$$

when the measurement is performed on the subsystem A . In the same manner, performing measurement on the second subsystem B , one gets

$$D_{AB}^{\leftarrow} = S_B - S_{AB} + E_{AC}. \quad (24)$$

It is simple to check that for a pure density state ρ_{AB} , the quantum discord reduces to entanglement of formation given by the entropy of the reduced density of the subsystem A .

3.2 Quantum discord in pure tripartite nonorthogonal states

In the pure bi-partitioning scheme (4), using the Wootters concurrence formula (21), it is simply verified that

$$\mathcal{C}_{k(ij)} = \frac{\sqrt{(1-p_k^2)(1-p_i^2 p_j^2)}}{1 + p_1 p_2 p_3 \cos m\pi}. \quad (25)$$

It follows that the entanglement of formation writes

$$E_{k(ij)} = H\left(\frac{1}{2} + \frac{1}{2} \frac{p_k + p_i p_j \cos m\pi}{1 + p_1 p_2 p_3 \cos m\pi}\right) \quad (26)$$

and coincides with the quantum discord

$$E_{k(ij)} = D_{k(ij)}. \quad (27)$$

For the mixed states ρ_{ij} associated with the second partitioning scheme (9), the concurrence (21) takes the following form

$$\mathcal{C}_{ij} = q_{ij} \frac{\sqrt{(1-p_i^2)(1-p_j^2)}}{1+p_1p_2p_3\cos m\pi}. \quad (28)$$

The pairwise quantum discord present in the mixed states ρ_{ij} can be computed using the procedure presented in the previous subsection. As result, when the measurement is performed on the subsystem $A \equiv i$, the quantum discord is

$$D_{ij}^{\rightarrow} = S_i - S_{ij} + E_{jk} \quad (29)$$

where k stands for the third subsystem traced out to get the reduced matrix density ρ_{ij} . The von Neumann entropy of the reduced density ρ_i is

$$S_i = H\left(\frac{1}{2} \frac{(1+p_i)(1+p_jq_{ij}\cos m\pi)}{1+p_1p_2p_3\cos m\pi}\right), \quad (30)$$

and the entropy of the bipartite density ρ_{ij} is explicitly given by

$$S_{ij} = H\left(\frac{1}{2} \frac{(1+p_ip_j\cos m\pi)(1+q_{ij})}{1+p_1p_2p_3\cos m\pi}\right). \quad (31)$$

It is important to emphasize that the entanglement of formation measuring the entanglement of the subsystem j with the ancillary qubit, required in the purification process to minimize the conditional entropy, is exactly the entanglement of formation measuring the degree of intricacy between the subsystem j and the traced out qubit k . It is given by

$$E_{jk} = H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{p_i^2(1-p_j^2)(1-p_k^2)}{(1+p_1p_2p_3\cos m\pi)^2}}\right). \quad (32)$$

Using the equations (30), (31) and (32), one obtains

$$D_{ij}^{\rightarrow} = H\left(\frac{(1+p_i)(1+p_jq_{ij}\cos m\pi)}{2(1+p_1p_2p_3\cos m\pi)}\right) - H\left(\frac{(1+p_ip_j)(1+q_{ij}\cos m\pi)}{2(1+p_1p_2p_3\cos m\pi)}\right) + H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{p_i^2(1-p_j^2)(1-q_{ij}^2)}{(1+p_1p_2p_3\cos m\pi)^2}}\right), \quad (33)$$

Also, because the whole system is pure, we have

$$S_{ij} = S_k \quad i, j \neq k. \quad (34)$$

Using the equations (30), (31) and (32), one obtains the following conservation relation

$$D_{12}^{\rightarrow} + D_{23}^{\rightarrow} + D_{31}^{\rightarrow} = E_{12} + E_{13} + E_{23},$$

reflecting that the sum of the bipartite quantum discord present in all mixed states ρ_{ij} is exactly the sum of the bipartite entanglement of formation. It is important to notice that the conservation law for

the distribution of entanglement of formation and quantum discord, in a pure tripartite system, was firstly derived in [51, 52]. Similarly, the explicit form of the quantum, when performing a measurement on the qubit j , is

$$D_{ij}^{\leftarrow} = S_j - S_{ij} + E_{ik}$$

and we have the following asymmetric relation

$$D_{ij}^{\leftarrow} = D_{ji}^{\rightarrow}. \quad (35)$$

The quantum discord D_{ij}^{\leftarrow} (resp. D_{ij}^{\rightarrow}) is the portion of the mutual information in the bipartite state ρ_{ij} that is locally inaccessible by i (resp. j). In this sense quantum discord can be interpreted as the fraction of the pairwise mutual information which can not be accessible by a local measurement. Based on the asymmetry definition of quantum discord, two useful quantities can be introduced: [52]

$$\Delta_{ij}^+ = \frac{1}{2}(D_{ij}^{\rightarrow} + D_{ij}^{\leftarrow}) \quad \Delta_{ij}^- = \frac{1}{2}(D_{ij}^{\rightarrow} - D_{ij}^{\leftarrow}).$$

The sum Δ_{ij}^+ is the average of locally inaccessible information when the measurements are performed on the subsystems i and j . It quantifies the disturbance caused by any local measurement. The difference Δ_{ij}^- was termed by Fanchini et al [52] the balance of locally inaccessible information and quantifies the asymmetry between the subsystems in responding to the measurement disturbance. Using the expressions of quantum discord given by (33) and the asymmetric relation (35), one verifies that the quantities Δ_{ij}^+ and Δ_{ij}^- satisfy the following distribution relations

$$\Delta_{12}^+ + \Delta_{13}^+ + \Delta_{23}^+ = E_{12} + E_{13} + E_{23}, \quad (36)$$

and

$$\Delta_{12}^- + \Delta_{13}^- + \Delta_{23}^- = 0. \quad (37)$$

Consequently, using the results (27) and (36), the global quantum correlation (1) when bipartite correlations are measured by quantum discord writes

$$D_{(1,2,3)} = \frac{1}{6} \left(E_{12} + E_{13} + E_{23} + E_{1(23)} + E_{2(13)} + E_{3(12)} \right). \quad (38)$$

This shows that the sum of quantum discord for all possible partitions coincides the global entanglement of formation

$$D_{(1,2,3)} = E_{(1,2,3)}. \quad (39)$$

4 Geometric quantum discord in tripartite nonorthogonal state

4.1 Definition

The geometric measure of quantum discord is defined as the distance between a state ρ of a bipartite system AB and the closest classical-quantum state presenting zero discord [20]:

$$D^g(\rho) := \min_{\chi} \|\rho - \chi\|^2 \quad (40)$$

where the minimum is over the set of zero-discord states χ and the distance is the square norm in the Hilbert-Schmidt space. It is given by

$$\|\rho - \chi\|^2 := \text{Tr}(\rho - \chi)^2.$$

When the measurement is taken on the subsystem A , the zero-discord state χ is represented as [11]

$$\chi = \sum_{i=1,2} p_i |\psi_i\rangle\langle\psi_i| \otimes \rho_i$$

where p_i is a probability distribution, ρ_i is the marginal density matrix of B and $\{|\psi_1\rangle, |\psi_2\rangle\}$ is an arbitrary orthonormal vector set. An arbitrary two qubit state writes in Bloch representation as

$$\rho = \frac{1}{4} \left[\sigma_0 \otimes \sigma_0 + \sum_i^3 (x_i \sigma_i \otimes \sigma_0 + y_i \sigma_0 \otimes \sigma_i) + \sum_{i,j=1}^3 R_{ij} \sigma_i \otimes \sigma_j \right] \quad (41)$$

where $x_i = \text{Tr}(\rho \sigma_i \otimes \sigma_0)$, $y_i = \text{Tr}(\rho \sigma_0 \otimes \sigma_i)$ are the components of local Bloch vectors and $R_{ij} = \text{Tr}(\rho \sigma_i \otimes \sigma_j)$ are components of the correlation tensor. The operators σ_i ($i = 1, 2, 3$) stand for the three Pauli matrices and σ_0 is the identity matrix. The explicit expression of the geometric quantum discord is given by [20]:

$$D^g(\rho) = \frac{1}{4} (\|x\|^2 + \|R\|^2 - k_{\max}) \quad (42)$$

where $x = (x_1, x_2, x_3)^T$, R is the matrix with elements R_{ij} and k_{\max} is the largest eigenvalue of the matrix defined by

$$K := xx^T + RR^T. \quad (43)$$

Denoting the eigenvalues of the 3×3 matrix K by λ_1 , λ_2 and λ_3 and considering $\|x\|^2 + \|R\|^2 = \text{Tr}K$, we get an alternative compact form of the geometric measure of quantum discord [45]

$$D^g(\rho) = \frac{1}{4} \min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\} \quad (44)$$

which is more convenient for our purpose.

4.2 Geometric measure of quantum discord for the pure bipartite states

Using the tools presented in the previous subsection, we shall determine the global geometric quantum discord in the tripartite state (3). We evaluate first the pairwise geometric discord in the pure bipartite states (4). For this, using the Schmidt decomposition, we write the state $|\Psi, m\rangle$ as

$$|\Psi, m\rangle = \sqrt{\lambda_+} |+\rangle_k \otimes |+\rangle_{ij} + \sqrt{\lambda_-} |-\rangle_k \otimes |-\rangle_{ij} \quad (45)$$

where $|\pm\rangle_k$ denotes the eigenvectors of the reduced density matrix associated with the first subsystem containing the particle k . Similarly, $|\pm\rangle_{ij}$ denotes the eigenvectors of the reduced density matrix for the second subsystem comprising the particles i and j . The eigenvalues λ_{\pm} are given by

$$\lambda_{\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 - \mathcal{C}_{k(ij)}^2} \right)$$

where the bipartite concurrence $\mathcal{C}_{k(ij)}$ is given by the equation (25). In this case, the matrix K , defined by (43), takes the diagonal form

$$K = \text{diag}(4\lambda_+\lambda_-, 4\lambda_+\lambda_-, 2(\lambda_+^2 + \lambda_-^2)),$$

and using the equation (44), the pairwise geometric discord is given by

$$D_{k(ij)}^g = \frac{1}{2} \frac{(1 - p_k^2)(1 - p_i^2 p_j^2)}{(1 + p_1 p_2 p_3 \cos m\pi)^2} \quad (46)$$

It is remarkable that the geometric quantum discord can be re-expressed as

$$D_{k(ij)}^g = \frac{1}{2} \mathcal{C}_{k(ij)}^2 \quad (47)$$

in terms of the bipartite concurrence $\mathcal{C}_{k(ij)}$. This equation traduces the relation between the geometric discord and the concurrence for pure bipartite states.

4.3 Geometric measure of quantum discord for mixed bipartite states

Having derived the geometric discord in the pure bipartition scheme, we now consider the mixed states of the form (9) obtained in the second bipartition scheme. In this order, we write the matrix ρ_{ij} as follows

$$\rho_{ij} = \sum_{\alpha\beta} R_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta \quad (48)$$

where the non vanishing correlation matrix elements $R_{\alpha\beta}$ ($\alpha, \beta = 0, 1, 2, 3$) are given by

$$R_{00} = 1, \quad R_{11} = 2\mathcal{N}^2 \sqrt{(1 - p_i^2)(1 - p_j^2)}, \quad R_{22} = -2\mathcal{N}^2 \sqrt{(1 - p_i^2)(1 - p_j^2)} p_k \cos m\pi,$$

$$R_{33} = 2\mathcal{N}^2(p_i p_j + p_k \cos m\pi), \quad R_{03} = 2\mathcal{N}^2(p_j + p_i p_k \cos m\pi), \quad R_{30} = 2\mathcal{N}^2(p_i + p_j p_k \cos m\pi).$$

In this case, the eigenvalues of the matrix K (43) write

$$\lambda_1 = 4\mathcal{N}^4 \left[(1 + p_i^2)(p_j^2 + p_k^2) + 4(p_1 p_2 p_3) \cos m\pi \right] \quad (49)$$

$$\lambda_2 = 4\mathcal{N}^4 (1 - p_i^2)(1 - p_j^2) \quad (50)$$

$$\lambda_3 = 4\mathcal{N}^4 (1 - p_i^2)(1 - p_j^2) p_k^2 \quad (51)$$

Noticing that $0 \leq p_i \leq 1$, it is easy to see that $\lambda_3 \leq \lambda_2$. Thus, the equation (44) reduces to

$$D_{ij}^g = \frac{1}{4} \min\{\lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}. \quad (52)$$

Subsequently, for the mixed states ρ_{ij} , the explicit expression of geometric quantum discord writes

$$D_{ij}^g = \frac{1}{4} \frac{(1 - p_i^2)(1 - p_j^2)(1 + p_k^2)}{(1 + p_1 p_2 p_3 \cos m\pi)^2} \quad (53)$$

when the condition $\lambda_1 > \lambda_2$ is satisfied or

$$D_{ij}^g = \frac{1}{4} \frac{(1 + p_i^2)(p_j^2 + p_k^2) + (1 - p_i^2)(1 - p_j^2)p_k^2 + 4(p_1 p_2 p_3) \cos m\pi}{(1 + p_1 p_2 p_3 \cos m\pi)^2} \quad (54)$$

in the situation where $\lambda_1 < \lambda_2$.

Finally the measure of multipartite quantum correlation (1) for geometric quantum discord, in the pure tripartite state (3), writes

$$D_{(1,2,3)}^g = \frac{1}{6} \left(D_{12}^g + D_{21}^g + D_{13}^g + D_{31}^g + D_{23}^g + D_{32}^g \right) + \frac{1}{12} \left(\mathcal{C}_{1(23)}^2 + \mathcal{C}_{2(13)}^2 + \mathcal{C}_{3(12)}^2 \right). \quad (55)$$

5 Illustration: three-mode Schrödinger cat states

To illustrate the results obtained in the previous sections, we need to consider a specific instance of tripartite system involving non orthogonal states. In this sense, we consider a three-mode Schrödinger cat state

$$|\alpha, m\rangle = \mathcal{N}_m(|\alpha|) \left(|\alpha\rangle_1 |\alpha\rangle_2 |\alpha\rangle_3 + e^{im\pi} |-\alpha\rangle_1 |-\alpha\rangle_2 |-\alpha\rangle_3 \right), \quad (56)$$

based on Glauber or radiation field coherent states $|\alpha\rangle$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (57)$$

where the complex number α characterizes the amplitude of the coherent state $|\alpha\rangle$ and $|n\rangle$ is a Fock state (also known as a number state). The normalization factor in (56) is given by

$$\mathcal{N}_m(|\alpha|) = (2 + 2e^{-6|\alpha|^2} \cos m\pi)^{-\frac{1}{2}}.$$

Considering this special tripartite state involving Glauber coherent states, we shall in what follows give the global quantum correlations $Q_{(1,2,3)}$ (see eq.(1)) when the pairwise correlations are measured by the squared concurrence, entanglement of formation, entropy based quantum discord or its geometrized variant. Furthermore, this specific tripartite state allows us to decide about the monogamy of each of these measures.

Two interesting limits of the Schrödinger cat states (56) arise when $\alpha \rightarrow \infty$ and $\alpha \rightarrow 0$. We first consider the asymptotic limit $\alpha \rightarrow \infty$. In this limit the two states $|\alpha\rangle$ and $|-\alpha\rangle$ approach orthogonality, and an orthogonal basis can be constructed such that $|\mathbf{0}\rangle \equiv |\alpha\rangle$ and $|\mathbf{1}\rangle \equiv |-\alpha\rangle$. Thus, the state $|\alpha, m\rangle$ approaches a multipartite state of GHZ type

$$|\alpha, m\rangle \sim |\text{GHZ}\rangle_3 = \frac{1}{\sqrt{2}} (|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle + e^{im\pi} |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle). \quad (58)$$

In the situation where $\alpha \rightarrow 0$, one should distinguish separately the cases $m = 0 \pmod{2}$ and $m = 1 \pmod{2}$. For m even, the tripartite superposition (56) reduces to ground state

$$|0, 0 \pmod{2}\rangle \sim |0\rangle \otimes |0\rangle \otimes |0\rangle, \quad (59)$$

and for m odd, the state $|\alpha, 1 \pmod{2}\rangle$ reduces to a multipartite state of W type [53]

$$|0, 1 \pmod{2}\rangle \sim |\text{W}\rangle_3 = \frac{1}{\sqrt{3}} (|1\rangle \otimes |0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |0\rangle \otimes |0\rangle \otimes |1\rangle). \quad (60)$$

Here $|n\rangle$ ($n = 0, 1$) denote the Fock-Hilbert states.

It follows that the states $|\alpha, m = 0 \pmod{2}\rangle$ interpolate between states of GHZ type ($\alpha \rightarrow \infty$) and the separable state $|0\rangle \otimes |0\rangle \otimes |0\rangle$ ($\alpha \rightarrow 0$). In other hand, the states $|\alpha, m = 1 \pmod{2}\rangle$ may be viewed as interpolating between states of GHZ type ($\alpha \rightarrow \infty$) and states of W type ($\alpha \rightarrow 0$).

5.1 Global quantum correlations and monogamy relation

5.1.1 Squared concurrence

Using the equation (25) and noticing that the states $\rho_{1(23)}, \rho_{2(13)}$ and $\rho_{3(12)}$ are identical, it is simple to check that the concurrences in the pure bipartite splitting are all equals. Explicitly, they are given by

$$\mathcal{C}_{1(23)} = \mathcal{C}_{2(13)} = \mathcal{C}_{3(12)} = \frac{\sqrt{(1-p^2)(1-p^4)}}{1+p^3 \cos m\pi}. \quad (61)$$

where $p = \langle \alpha | -\alpha \rangle = e^{-2|\alpha|^2}$. In the the second bipartite splitting (8), the mixed density matrices ρ_{12}, ρ_{23} and ρ_{13} are identical and the concurrence (28) rewrites

$$\mathcal{C}_{12} = \mathcal{C}_{23} = \mathcal{C}_{13} = \frac{p(1-p^2)}{1+p^3 \cos m\pi}. \quad (62)$$

To examine the monogamy relation of entanglement measured by the concurrence in quantum systems involving three qubits, Coffman et al [33] introduced the so called three tangle defined as follows

$$\tau_{i|jk} = \mathcal{C}_{i(jk)}^2 - \mathcal{C}_{ij}^2 - \mathcal{C}_{ik}^2. \quad (63)$$

Reporting (61) and (62) in (63), one gets

$$\tau_{1|23} = \tau_{2|13} = \tau_{3|12} \equiv \tau$$

with

$$\tau = \frac{(1-p^2)^2(1-p)^2}{(1+p^3 \cos m\pi)^2}.$$

The three tangle τ is always positive. This result reflects the monogamy of entanglement measured by the squared concurrence. In other hand, using the expressions (61) and (62) and replacing the pairwise quantum correlation Q in (1) by the squared concurrence, the global tripartite quantum correlation (1) in the tripartite Schrödinger cat states (56) takes the following form

$$\mathcal{C}_{(1,2,3)}^2 = \frac{1}{2} \frac{(1+2p^2)(1-p^2)^2}{(1+p^3 \cos m\pi)^2}.$$

5.1.2 Entanglement of formation and quantum discord

As above, to decide about the monogamy of entanglement measured by the entanglement of formation, we introduce the following quantity

$$E_{i|jk} = E_{i(jk)} - E_{ij} - E_{ik}. \quad (64)$$

For the Schrödinger cat states under consideration, the pairwise entanglement of formation corresponding to the pure bipartition scheme (4) can be obtained from equation (26). One gets

$$E_{1(23)} = E_{2(13)} = E_{3(12)} = H\left(\frac{1}{2} + \frac{1}{2} \frac{p + p^2 \cos m\pi}{1 + p^3 \cos m\pi}\right) \quad (65)$$

In the second splitting scheme (8), we have $\rho_{12} = \rho_{23} = \rho_{13}$. In this case, the equation (32) gives

$$E_{12} = E_{23} = E_{13} = H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{p^2(1-p^2)^2}{(1+p^3 \cos m\pi)^2}}\right). \quad (66)$$

Substituting the expressions (65) and (66) in the equation (64), one obtains

$$E_{1|23} = E_{2|13} = E_{3|12} \equiv E$$

where the quantity E is given by

$$E = H\left(\frac{1}{2} + \frac{1}{2} \frac{p + p^2 \cos m\pi}{1 + p^3 \cos m\pi}\right) - 2H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{p^2(1-p^2)^2}{(1+p^3 \cos m\pi)^2}}\right).$$

The behavior of the quantity E vs the overlap p is depicted in the following figure.

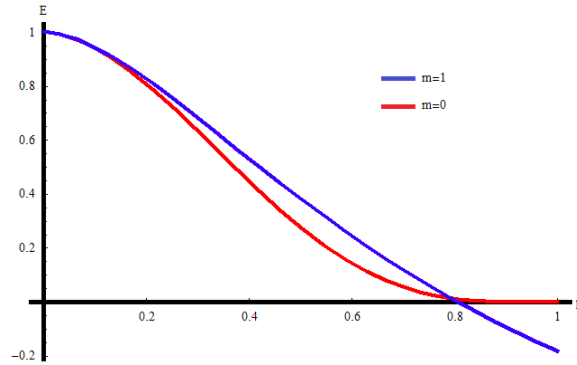


Figure 1 $E = E_{i|jk}$ versus the overlapping p for $m = 0$ and $m = 1$.

Clearly, the entanglement of formation is monogamous for symmetric three modes Schrödinger cat states ($m = 0$) for any value of p . The antisymmetric states ($m = 1$) possess monogamy property only when $0 \leq p \lesssim 0.8$. The figure 3 reveals that the $|GHZ\rangle_3$ state ($p \rightarrow 0$) follows monogamy and $|W\rangle_3$ state ($p \rightarrow 1$) does not.

The sum of the pairwise entanglement of formation, in all possible bi-partitions, is then given by

$$E_{(1,2,3)} = \frac{1}{2} \left[H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{p^2(1-p^2)^2}{(1+p^3 \cos m\pi)^2}}\right) + H\left(\frac{1}{2} + \frac{1}{2} \frac{p + p^2 \cos m\pi}{1 + p^3 \cos m\pi}\right) \right]. \quad (67)$$

To compute the global amount of pairwise quantum discord in the states (56) and to investigate the monogamy relation, two important remarks are in order. First, note that in a pure state the entanglement of formation and quantum discord coincide. In this respect, in the pure bipartition scheme (4), one has

$$E_{1|23} = D_{1|23} \quad E_{2|13} = D_{2|13} \quad E_{3|12} = D_{3|12}$$

Furthermore, using the equations (32) and (33), one can verify that for the reduced mixed states $\rho_{12} = \rho_{13} = \rho_{23}$, the entanglement of formation coincides with quantum discord. Indeed, we have

$$E_{12} = D_{12} \quad E_{23} = D_{23} \quad E_{13} = D_{13}.$$

It is remarkable that the bipartite mixed states ρ_{12} , ρ_{13} and ρ_{23} constitute a special class of mixed states where entanglement of formation coincides with quantum discord. Thus, the measures of entanglement of formation and quantum discord, in the Schrödinger cat states (56), are identical and the global amount of quantum discord coincides, as expected, with the global entanglement of formation given by (67).

5.1.3 Geometric quantum discord

Now, we consider the global quantum correlation measured by geometric quantum discord. For the states (56), from the equation (46), one has

$$D_{1(23)}^g = D_{2(13)}^g = D_{3(12)}^g$$

with

$$D_{1(23)}^g = \frac{1}{2} \mathcal{C}_{1(23)}^2 = \frac{1}{2} \frac{(1-p^2)(1-p^4)}{(1+p^3 \cos m\pi)^2}. \quad (68)$$

For the mixed states ρ_{12} , ρ_{13} and ρ_{23} which are identical, we treat the symmetric and anti-symmetric cases separately. For $m = 0$, using (53), the geometric quantum discord writes

$$D_{12}^g = D_{23}^g = D_{13}^g = \frac{1}{4} \frac{p^2(1+p)^2(2+(1-p)^2)}{(1+p^3)^2} \quad (69)$$

for $0 \leq p \leq \sqrt{2} - 1$ and from (54) one obtains

$$D_{12}^g = D_{23}^g = D_{13}^g = \frac{1}{4} \frac{(1+p^2)(1+p)^2(1-p)^2}{(1+p^3)^2} \quad (70)$$

when $\sqrt{2} - 1 \leq p \leq 1$. For the antisymmetric Schrödinger cat states ($m = 1$), the geometric quantum discord is

$$D_{12}^g = D_{23}^g = D_{13}^g = \frac{1}{4} \frac{p^2(2+(1+p)^2)}{(1+p+p^2)^2} \quad (71)$$

It follows that, for even tripartite Schrödinger cat states ($m = 0$), the total amount of quantum correlation measured by the geometric discord is

$$D_{(1,2,3)}^g = \frac{1}{8} \frac{(1+p)^2(2p^2+(1-p^2)(2+3p^2))}{(1+p^3)^2}$$

for $0 \leq p \leq \sqrt{2} - 1$, and

$$D_{(1,2,3)}^g = \frac{3}{8} \frac{(1+p^2)(1-p)^2}{(1+p^3)^2}$$

when $\sqrt{2} - 1 \leq p \leq 1$. For odd Schrödinger cat states ($m = 1$), the sum of all possible pairwise geometric quantum discord is given by the following equation

$$D_{(1,2,3)}^g = \frac{1}{8} \frac{2p^2 + (1+p)^2(2+3p^2)}{(1+p+p^2)^2}$$

for $0 \leq p \leq 1$.

Note that the maximal value of geometric discord (40) for two qubit states is $1/2$ and it is not normalized to one. Hence, for comparison with the others normalized measures, we consider $2D^g$ as a proper measure.

In the figures 2 and 3, a comparison of tripartite quantum correlation for the squared concurrence, usual quantum discord and its geometrized version are represented. Figure 2 displays that these three measures give approximatively the same amount of quantum correlation for $m = 0$. This corroborates the fact that the entanglement of formation, quantum discord and geometric quantum discord possess the monogamy property like the squared concurrence. Figure 3 reveals that for $m = 1$, the sum of entanglement of formation (or equivalently the usual quantum discord) becomes larger than the sum of pairwise quantum correlations measured by the concurrence and the geometric discord, especially when p approaches the unity. Furthermore, the global sum of squared concurrences behaves like the sum of bipartite geometric discord for $0 \leq p \leq 0.5$ and increases slowly after but the behavior stays slightly the same as geometric discord.

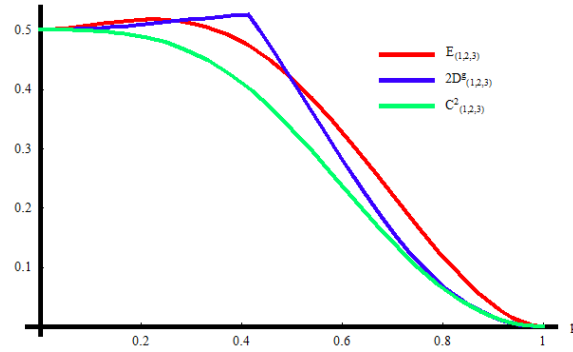


Figure 2 Tripartite quantum correlation versus the overlapping p for $m = 0$.

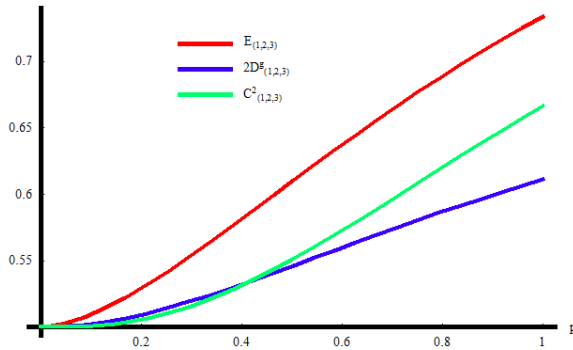


Figure 3 Tripartite quantum correlation versus the overlapping p for $m = 1$.

Finally, to examine the monogamy of geometric quantum discord, one should analyze the positivity

of the following quantity

$$D_{i|jk}^g = D_{i(jk)}^g - D_{ij}^g - D_{ik}^g.$$

For the tripartite cat states (56), we have

$$D_{1|23}^g = D_{2|13}^g = D_{3|12}^g \equiv D^g.$$

In the symmetric case ($m = 0$), the quantity D^g vanishes for $\sqrt{2} - 1 \leq p \leq 1$ and it is given by

$$D^g = \frac{1}{2} \frac{(1+p)^2(1-(\sqrt{2}+1)p)(1-(\sqrt{2}-1)p)}{(1+p^3)^2}$$

for $0 \leq p \leq \sqrt{2} - 1$. It is simple to verify that in this case the geometric discord is monogamous. For antisymmetric Schrödinger cat states ($m = 1$), one obtains

$$D^g = \frac{1}{2} \frac{(1+2p-p^2)}{(1+p+p^2)^2},$$

which is always positive. In this respect, The geometric quantum discord follows the monogamy property for any value of the overlap p .

6 Concluding remarks

In summary, we have explicitly derived the quantum correlation in a tripartite system involving nonorthogonal states. The total amount of quantum correlation is defined as the sum of all pairwise quantum correlations. It is evaluated using measures which go beyond entanglement, e.g., usual quantum discord and its geometrized version. A suitable qubit mapping was realized for all possible bi-partitions of the system. We have shown that the sum of all pairwise entanglement of formation in a pure entangled tripartite state is exactly the sum of pairwise quantum discord of all possible bi-partitions. This peculiar result originates from the conservation relation between the entanglement of formation and quantum discord. We also examined the monogamy relation of concurrence, entanglement of formation, quantum discord and quantum discord in the special case of non orthogonal three-modes Schrödinger cat states. We proved that squared concurrence and geometric discord are monogamous. The entanglement of formation and quantum discord follows the monogamy property in the symmetric tripartite Schrödinger cat states ($m = 0$). However, in the antisymmetric case ($m = 1$), they cease to be monogamous when the three-mode cat states approach the three qubit states W_3 corresponding to the situation where $p \rightarrow 1$. The odd Schrödinger cat states (56) interpolate continuously between the GHZ type states (58) ($p \rightarrow 0$) and W states (60) ($p \rightarrow 1$). The GHZ states maximize the pure entanglement of formation $E_1(23)$ between any qubit and the two others. The W states maximize the entanglement of formation E_{12} in the mixed states obtained after tracing out the third qubit.

Finally, It must be noticed that the investigation of monogamy and polygamy of quantum correlations in multipartite quantum systems is deeply dependent on the choice of correlations measures.

Many exciting issues, regarding this problem, remain open. The quantification of the genuine multipartite correlations constitutes a key challenge in the field of quantum information theory to understand the distribution of correlations in quantum systems comprising many parts.

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